11/08/21

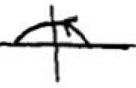
§ 16.2-16.3 Line Integrals

IDEA: Given a curve $C: [a,b] \to D \subseteq \mathbb{R}^n$ and a function $f: D \to \mathbb{R}$. How does f "behave" along the curve? i.e. what does f "contribute" to the curve?

Def^n: The line integral (or path integral) of function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ along curve C parameterized by $\vec{r}: [a,b] \to D$ is $\int_C f\, ds = \int_{t=a}^{b} f(\vec{r}(t)) |\vec{r}\,'(t)|\, dt$

Remark: If $f(\vec{r}) = 1$ for all $\vec{r}$, then $\int_C 1\, ds = \int_{t=a}^{b} |\vec{r}\,'(t)|\, dt = s(C)$
(Arc length of C)

Ex: Compute $\int_C f\, ds$ for $f(x,y) = 2 + x^2 y$ along C, the upper half of the unit circle w/ positive orientation.



Sol: $\int_C (2 + x^2 y)\, ds$

C is parameterized by
$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le \pi$
$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$
$|\vec{r}\,'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$

$$= \int_{t=0}^{\pi} (2 + \cos^2(t)\sin(t)) \cdot 1\, dt$$

$$= \int_0^{\pi} 2\, dt + \int_0^{\pi} \cos^2(t)\sin(t)\, dt$$

$$= 2[t]_0^{\pi} - \int_0^{\pi} u^2\, du = 2(\pi - 0) - \tfrac{1}{3}[u^3]_0^{\pi}$$

$$= 2\pi - \tfrac{1}{3}[\cos^3(t)]_0^{\pi} = 2\pi - \tfrac{1}{3}((-1)^3 - 1^3) = \boxed{2\left(\pi + \tfrac{1}{3}\right)}$$

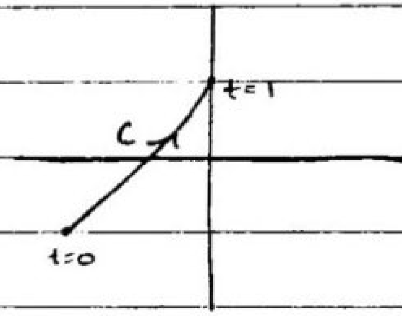
To measure the "buildup" of f-values in one direction x_k, we can see

$$\int_C f\, dx_k = \int_{t=a}^{b} f(\vec{r}(t))\, |x_k(t)|\, dt$$

where $x_k(t)$ is the x_k-component of $\vec{r}(t)$ a parameterization of C.

Ex: Evaluate $\int_C y^2\,dx + \int_C x\,dy$ where C is the line segment oriented from $(-5,-3)$ to $(0,2)$

Sol: First parameterize C.



$$\vec{r}(t) = (1-t)\langle -5,3\rangle + t\langle 0,2\rangle$$
$$= \langle -5+5t, -3+3t+2t\rangle$$
$$= \langle -5+5t, -3+5t\rangle$$

$$\vec{r}\,'(t) = \langle \underbrace{5}_{x'(t)}, \underbrace{5}_{y'(t)}\rangle$$

$$\therefore \int_C y^2\,dx + \int_C x\,dy$$

$$= \int_{t=0}^{1} (5t-3)^2 \cdot 5\,dt + \int_{t=0}^{1} (5t-5)\cdot 5\,dt$$

$$= \int_{t=0}^{1} 5(5t-3)^2 + 5(5t-5)\,dt = 5\int_{t=0}^{1} (25t^2-30t+9+5t-5)\,dt$$

$$= 5\int_0^1 (25t^2-25t+4)\,dt = 5\left[\tfrac{25}{3}t^3 - \tfrac{25}{2}t^2 + 4t\right]_0^1$$

$$= 5\left(\left(\tfrac{25}{3} - \tfrac{25}{2} + 4\right) - 0\right) = 5\left(-\tfrac{25}{6} + \tfrac{24}{6}\right) = 5\left(-\tfrac{1}{6}\right) = \boxed{-\tfrac{5}{6}}$$

Def^{n}: The line integral of vector field $\vec{v}$ along curve C parameterized by $\vec{r}(t)$ for $a \leq t \leq b$ is

$$\int_C \vec{v} \cdot d\vec{r} = \int_{t=a}^{b} \vec{v}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_C \vec{v} \cdot \vec{T}\,ds$$, where $\vec{T}$ is the unit tangent of $\vec{r}$, i.e. $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

Ex: Compute $\int_C \vec{v} \cdot dr$ for $\vec{v}(x,y,z) = \langle xy, yz, zx \rangle$ and c the curve parameterized by $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ on $0 \leq t \leq 2$.

Sol: $\int_C \vec{v} \cdot d\vec{r}$

$$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\vec{v}(\vec{r}(t)) = \langle t \cdot t^2, t^2 \cdot t^3, t^3 \cdot t \rangle = \langle t^3, t^5, t^4 \rangle$$

$$= \int_{t=a}^{b} \vec{v}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_{t=0}^{2} \langle t^3, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle\,dt$$

$$= \int_{t=0}^{2} (t^3 + 2t^6 + 3t^6)\,dt$$

$$= \int_{t=0}^{2} (t^3 + 5t^6)\,dt$$

$$= \left[\frac{1}{4}t^4 + \frac{5}{7}t^7\right]_{t=0}^{2} = \left(\frac{16}{4} + \frac{5(128)}{7}\right) - 0$$

$$= \frac{28 + 640}{7}$$

$$= \boxed{\frac{668}{7}}$$

NB: Physics work is just a line integral...
ie. the work done by a particle moving along path $\vec{r}(t)$ for $a \le t \le b$ through vector field $\vec{F}$ is $\int_C \vec{F} \cdot dr$.

Exercise: Compute the work done by particle taking path along the clockwise-oriented quarter circle from $(0,1)$ to $(1,0)$ mainly through vector field $\vec{F} = \langle x^2, -xy \rangle$

Think back to 2nd Example:

$$\int_C y^2 dx + \int_C x dy$$

we can abbreviate this type of line integral as

$$\int_C y^2 dx + x dy$$

← NB: requires integration along same curve

In general we abbreviate

$$\int_C P \cdot dx + Q dy = \int_C P \cdot dx + \int_C Q dy$$

Idea: Line integrals are one-dimensional that got twisted up in n-space

Q: Is there an analogue of the Fundamental Theorem of Calculus for Line Integrals?

Best Ans: Antiderivatives of $f: \mathbb{R}^n \to \mathbb{R}$ don't really make sense... So the answer must be "no" for general "scalar line integrals"

Good News: If $\vec{v}$ is a conservative vector field, then it's potential functions act like antiderivatives ...

So there is some hope for Conservative vector fields.

Prop (Fundamental Theorem of Line Integrals): ← FTLI

↳ If C is a smooth curve parameterized by $\vec{r}(t)$ on $[a,b]$ and $f: \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives on C, then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof: Using FTC and the multivariate chain rule:

$$\int_C \nabla f \, d\vec{r} = \int_{t=a}^{b} \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

By multivariate chain rule $= \int_{t=a}^{b} \frac{d}{dt}\left[f(\vec{r}(t))\right] dt$

By FTC $= f(\vec{r}(b)) - f(\vec{r}(a))$

Ex: Compute $\int_C \vec{v} \cdot d\vec{r}$ via the FTLI for $\vec{v} = \langle 3 + 2xy^2, 2x^2y \rangle$ on $\vec{r}(t) = \langle t, \frac{1}{t} \rangle$ for $1 \le t \le 4$

Sol: First compute a potential:

$$f(x,y) = \int \frac{\partial F}{\partial x} dx = \int (3 + 2xy^2) dx = 3x + x^2y^2 + C(y)$$

$\because\; 2x^2y = \frac{\partial f}{\partial y}$

$= \frac{\partial}{\partial y}\left[3x + x^2y^2 + c(y)\right]$

$= 2x^2y + c'(y) \quad \rightarrow \quad c'(y) = 0$

$\therefore C(y) = D$ for some constant D.

$f(x,y) = 3x + x^2y^2 + D$ is a potential for $\vec{v}$ for all D, in particular, $D=0$ works and $\nabla(3 + x^2y^2) = \vec{v}$.

$$\begin{aligned}\int_C \vec{v}\, dr &= f(\vec{r}(4)) - f(\vec{r}(1)) \\ &= f(4,\tfrac{1}{4}) - f(1,1) \\ &= 3\cdot 4 + 4^2(\tfrac{1}{4})^2 - (3(1) + 1^2 + 1^2) \\ &= 9\end{aligned}$$